Total marks – 75

Attempt Questions 1-4

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

QUESTION 1 *(22 marks)* ***START A NEW BOOKLET*** **Marks**

(a) Evaluate $\int \frac{\cos x}{\sin x} dx$. **2**

(b) Consider the curves $y = e^x$ and $y = x^3 - 3x + 1$.

(i) Show that they have a common point at $(0,1)$. **1**

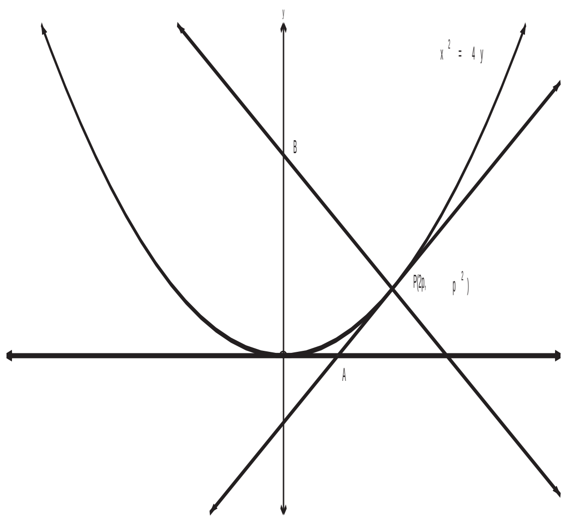
(ii) Find the acute angle (to the nearest minute) between the tangents to the curves at this point. **3**

(c) Use $2\cos^2\theta = 1 + \cos 2\theta$ to prove that $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$. **3**

(d) Let $f(x) = \ln(\tan x)$, $0 < x < \frac{\pi}{2}$. Show that $f'(x) = 2\text{cosec}2x$. **2**

(e) Use mathematical induction to prove that $3^{2n-1} + 5$ is divisible by 8, for all integers $n \geq 1$. **5**

(f) The diagram shows a point $P(2p, p^2)$ on the parabola $x^2 = 4y$. The tangent to the parabola at P cuts the *x*-axis at A. The normal to the parabola at point P cuts the y-axis at B.



(i) Find the equation of the tangent AP. **1**

(ii) Show that B has coordinate $(0, p^2 + 2)$ **2**

(iii) Let M be the midpoint of AB. Find the Cartesian equation of the locus of M. **3**

QUESTION 2 *(18 marks)* ***START A NEW BOOKLET*** **Marks**

(a) Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{4 - x^2}}\,dx$ **2**

(b) (i) Differentiate $\sin^{-1} x^3$ with respect to x. **2**

(ii) Hence or otherwise find $\displaystyle\int \frac{2x^2}{\sqrt{1 - x^6}}\,dx$ **1**

(c) Consider the function $f(x) = 3\sin^{-1}(2x - 1)$

(i) Find the domain of $f(x)$ **2**

(ii) Find the range of $f(x)$ **2**

(iii) Sketch the graph of $f(x)$ **2**

(d) Consider the function $f(x) = e^{-x} - e^{x}$

(i) Show that $f(x)$ is decreasing for all values of x. **2**

(ii) Show that the inverse function is given by:

$$f^{-1}(x) = \log_e\left(\frac{-x + \sqrt{x^2 + 4}}{2}\right)$$ **3**

(iii) Hence solve $e^{-x} - e^{x} = 6$, giving your answer in simplest surd form. **2**

QUESTION 3 *(18 marks)* ***START A NEW BOOKLET*** **Marks**

(a) Solve $\sin^2\theta = \frac{1}{2}\sin 2\theta$, $0 \le \theta \le 2\pi$ **3**

(b) (i) Express $y = \sqrt{3}\sin x + \cos x$ in the form $R\sin(x + \alpha)$, $R > 0$ and α is acute. **3**

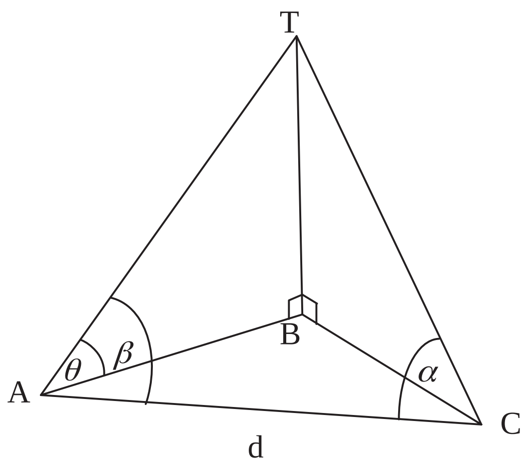
(ii) Find the maximum and minimum values of $y = \sqrt{3}\sin x + \cos x$ **1**

(iii) Solve $\sqrt{3}\sin x + \cos x = 1$, $0 \le x \le 2\pi$ **2**

(c) Evaluate $\int_0^{\frac{\pi}{4}} \sin^2(\frac{1}{2}x)\,dx$ **3**

(d) Using the substitution $t = \tan\frac{\theta}{2}$ prove that $\frac{1 - \cos\theta}{1 + \cos\theta} = \tan^2\frac{\theta}{2}$ **3**

(e) TB is a tower such that the angle of elevation of T, the top of the tower, from point A is θ. A, B and C are on the same level and $\angle CAT = \beta$ and $\angle ACT = \alpha$. If $AC = d$ metres then show that the height of the tower is given by: $TB = \frac{d\sin\theta\sin\alpha}{\sin(\alpha + \beta)}$. **3**



QUESTION 4 *(17 marks)* ***START A NEW BOOKLET*** **Marks**

(a) The velocity v, in metres per second, of a particle travelling in a straight line is given by $v = 2t - 4$. The initial displacement of the particle is given by $x = 3$.

(i) Find the acceleration of the particle. **1**

(ii) Find the displacement of the particle when it is stationary. **3**

(iii) How far does the particle travel in the first 3 seconds. **1**

(b) A bug is oscillating in simple harmonic motion such that its displacement x metres from a fixed point O at time t seconds is given by the equation $\ddot{x} = -4x$.
When $t = 0, v = 2m/s$ and $x = 5$.

(i) Show that $x = a\cos(2t + \alpha)$ is a solution for this equation, where a and α are constants. **1**

(ii) Find the period of the motion. **1**

(iii) Show that the amplitude of the oscillation is $\sqrt{26}$. **3**

(iv) What is the maximum speed of the bug? **2**

(c) The acceleration of a particle P is given by the equation

$$\frac{d^2x}{dt^2} = 8x(x^2 + 4)$$

where x metres is the displacement of P from a fixed point O after t seconds.

Initially the particle is at O and has velocity $8ms^{-1}$ in the positive direction.

(i) Show that the speed at any position x is given by $2\left(x^2 + 4\right)$ **3**

(ii) Hence find the time taken for the particle to travel 2 metres from O. **2**

END OF THE PAPER

ion 1 Yr12 Ext 1 Mini 2011

(a) $\int \frac{\cos x}{\sin x}\,dx = \ln\sin x + C$ ✓✓

(b) (i) $y = e^x$: $x = 0$, $y = 1$

$y = x^3 - 3x + 1$: $x = 0$, $y = 1$ ✓

(ii) $m_1 = e^0 = 1$

$m_2 = 3x^2 - 3 =$

$= -3$ ✓

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{1+3}{1-3}\right|$ ✓

$= \left|\frac{4}{-2}\right|$

$= 2$

$\therefore \theta = \tan^{-1} 2$

$= 63°26'$ (nearest minute) ✓

(c) $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

$\cos\theta = \pm\sqrt{\frac{1 + \cos 2\theta}{2}}$ ✓

$\therefore \cos\frac{\pi}{8} = \sqrt{\frac{1 + \cos\frac{\pi}{4}}{2}}$ $\left(\cos\frac{\pi}{8} > 0\right)$ ✓ Must discuss + and −

$= \sqrt{\frac{1 + \sqrt{2}/2}{2}}$ ✓

$= \sqrt{\frac{2 + \sqrt{2}}{4}}$

$= \frac{\sqrt{2 + \sqrt{2}}}{2}$

abd	8
c	3
ef	11

(d) $f(x) = \ln \tan x \qquad 0 < x < \frac{\pi}{2}$

$$f'(x) = \frac{\sec^2 x}{\tan x}$$

$$= \frac{1}{\sin x \cos x}$$

$$= \frac{2}{\sin 2x}$$

$$= 2\operatorname{cosec} 2x$$

(e) To prove: $3^{2n-1} + 5$ is divis. by 8 for all integers $n \geq 1$

$n=1$: $3+5=8$

$\therefore$ true for $n=1$

$n=k$: assume that it is true for $n=k$

$3^{2k-1} + 5 = 8M \qquad (M \in \mathbb{Z})$

$\therefore 3^{2k-1} = 8M - 5$

$n=k+1$: prove true for $n=k+1$

i.e. $3^{2(k+1)-1} + 5 = 8N \quad (N \in \mathbb{Z})$

$LHS = 3^{2k+1} + 5$

$= 3^2 \, 3^{2k-1} + 5$

$= 9(8M-5) + 5$

$= 72M - 40$

$= 8(9M-5)$

$= 8N \qquad$ (since $M \in \mathbb{Z}$ then $9M-5 \in \mathbb{Z}$)

$\therefore$ Since it is true for $n=1$, and if we assume that it is true for $n=k$ then it is also true for $n=k+1$, then by the principle of math. induction it is true for $n=1,2,3,\ldots$

(f)(i) $y - y_1 = m(x - x_1)$

$y - p^2 = p(x - 2p)$

$y - p^2 = px - 2p^2$

$px - y - p^2 = 0$ ✓

(ii) gradient of normal is $\frac{-1}{p}$

$y - p^2 = \frac{-1}{p}(x - 2p)$ ✓

$y - p^2 = \frac{-1}{p}x + 2$

$y = \frac{-1}{p}x + 2 + p^2$ ✓

when $x = 0$ $y = p^2 + 2$

$\therefore$ B is $(0, p^2+2)$

(iii) in $px - y - p^2 = 0$ let $y = 0$

$px - p^2 = 0$

$px = p^2$

$x = p$ ✓

$\therefore$ A is $(p, 0)$

$\therefore M = \left(\frac{0+p}{2}, \frac{p^2+2+0}{2}\right)$

$= \left(\frac{p}{2}, \frac{p^2+2}{2}\right)$ ✓

let $x = \frac{p}{2}$ $\therefore p = 2x$

$y = \frac{p^2+2}{2}$

$= \frac{(2x)^2+2}{2}$

$= \frac{4x^2+2}{2}$

$\therefore y = 2x^2 + 1$ ✓

2(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_0^1$ ✓

$= \sin^{-1}\frac{1}{2}$

$= \frac{\pi}{6}$ ✓

[2]

(b) (i) $\dfrac{d\left[\sin^{-1}x^3\right]}{dx} = \dfrac{3x^2}{\sqrt{1-x^6}}$ ✓✓

[3] (ii) $\displaystyle\int \frac{2x^2}{\sqrt{1-x^6}}\,dx = \frac{2}{3}\sin^{-1}x^3 + C$ ✓

(c) $f(x) = 3\sin^{-1}(2x-1)$

(i) Domain $-1 \le 2x-1 \le 1$ ✓

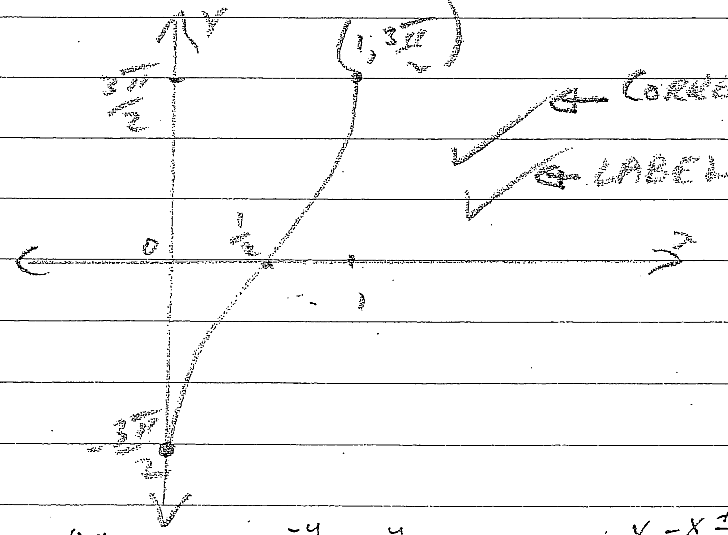
$0 \le 2x \le 2$

$0 \le x \le 1$ ✓

[6] (ii) Range $-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$ ✓

$-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ ✓

(iii)



✓ ← CORRECT POSITION

✓ ← LABELS (all)

(d)(i) $f(x) = e^{-x} - e^{x}$

$f'(x) = -e^{-x} - e^{x}$

$= -(e^{-x} + e^{x})$ ✓

< 0 as $e^{-x} > 0$ and $e^{x} > 0$ for all x ✓

(ii) let $x = e^{-y} - e^{y}$

$\therefore e^{y}x = 1 - e^{2y}$

$\therefore (e^{y})^2 + x(e^{y}) - 1 = 0$ ✓

$\therefore e^{y} = \dfrac{-x \pm \sqrt{x^2+4}}{2}$ ✓ [TAK +

$\therefore y = \ln\left[\dfrac{-x+\sqrt{x^2+4}}{2}\right]$ ✓

(iii) let $f^{-1}[f(x)] = x$

$\therefore x = \ln\left[\dfrac{-6+\sqrt{40}}{2}\right]$ ✓

$= \ln[-3+\sqrt{10}]$ ✓

Solutions — Q3

(a) $\sin^2\theta = \frac{1}{2}\sin 2\theta$, $0 \le \theta \le 2\pi$.

$\sin^2\theta = \frac{1}{2}(2\sin\theta\cos\theta)$ 1

$\therefore \sin^2\theta - \sin\theta\cos\theta = 0$.

$\therefore \sin\theta(\sin\theta - \cos\theta) = 0$

$\sin\theta = 0$ or $\sin\theta - \cos\theta = 0$

$\therefore \sin\theta = \cos\theta$; $\cos\theta \neq 0$.

$\therefore \theta = 0, \pi, 2\pi$ $\therefore \tan\theta = 1$

1 or $\theta = \pi/4, 5\pi/4$. 1

$\therefore \theta = 0, \pi/4, 5\pi/4, \pi, 2\pi$. (3)

(b) (i) $y = \sqrt{3}\sin x + \cos x$

let $\sqrt{3}\sin x + \cos x = R\sin(x+\alpha)$

$\therefore \sqrt{3}\sin x + \cos x = R\sin x\cos\alpha + R\cos x\sin\alpha$.

$\therefore R\cos\alpha = \sqrt{3}$ } 1

$R\sin\alpha = 1$ } 1

$\therefore R^2 = 4$ $\therefore R = 2$ 1 (3)

and $\tan\alpha = \frac{1}{\sqrt{3}}$ $\therefore \alpha = \pi/6$.

$\therefore \sqrt{3}\sin x + \cos x = 2\sin(x + \pi/6)$.

(ii) Maximum + Minimum : $-2 \le 2\sin(x+\pi/3) \le 2$.

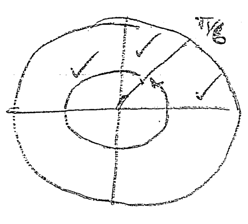
$\therefore$ Maximum = 2 and Minimum = −2. (1)

(iii) If $\sqrt{3}\sin x + \cos x = 1$; $0 \le x \le 2\pi$.

then $2(\sin(x+\pi/6)) = 1$; $\pi/6 \le x + \pi/3 \le 13\pi/6$

$\therefore \sin(x+\pi/6) = \frac{1}{2}$

If $\sin\alpha = \frac{1}{2}$, $\alpha = \pi/6$. 1



$\therefore x + \pi/6 = \pi/6$ or $x + \pi/6 = 5\pi/6$

$\therefore x = -\pi/6$ or $x = 2\pi/3$

invalid

or $x + \pi/6 = 13\pi/6$.

or $x = 2\pi$

$\therefore x = 0, \frac{2\pi}{3}, 2\pi$ (2)

Q3

(c) for $\int_0^{\pi/4} \sin^2\left(\frac{1}{2}x\right) dx$;

Now, $\cos 2\theta = 1 - 2\sin^2\theta$

$\therefore 2\sin^2\theta = 1 - \cos 2\theta$

$\therefore \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\therefore \sin^2\left(\frac{1}{2}x\right) = \frac{1}{2}(1 - \cos x)$. 1

$\therefore \int_0^{\pi/4} \sin^2\left(\frac{1}{2}x\right) dx = \frac{1}{2}\int_0^{\pi/4} 1 - \cos x \, dx$

$= \frac{1}{2}\left[x + \sin x\right]_0^{\pi/4}$. 1

$= \frac{1}{2}\left(\pi/4 + \frac{\sqrt{2}}{2}\right) - \frac{1}{2}(0)$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{\sqrt{2}}{2}\right)$. 1 (3)

(d) Let $t = \tan\frac{\theta}{2}$; $\frac{1 - \cos\theta}{1 + \cos\theta} = \tan^2\frac{\theta}{2}$.

Now $\cos\theta = \frac{1 - t^2}{1 + t^2}$ 1

$\therefore \frac{1 - \cos\theta}{1 + \cos\theta} = \left(1 - \frac{1 - t^2}{1 + t^2}\right) \div \left(1 + \frac{1 - t^2}{1 + t^2}\right)$

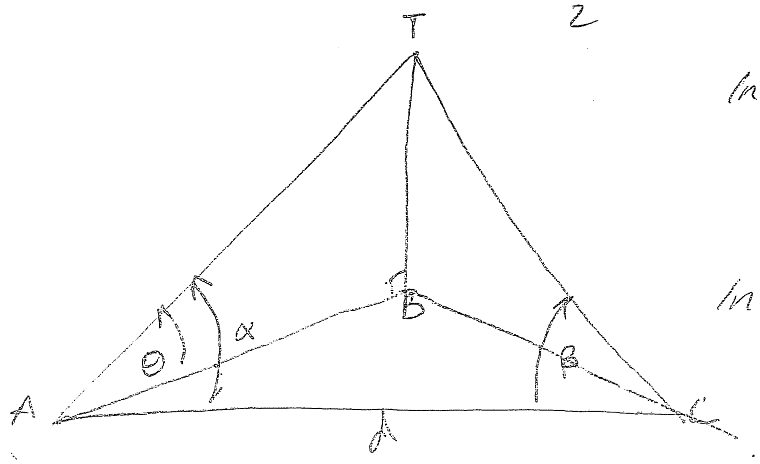
$= \left(\frac{1 + t^2 - 1 + t^2}{1 + t^2}\right) \div \left(\frac{1 + t^2 + 1 - t^2}{1 + t^2}\right)$. 1

$= \frac{2t^2}{1 + t^2} \times \frac{1 + t^2}{2}$

$= t^2$

$= \tan^2\frac{\theta}{2}$. 1 (3)

(e)



In ΔABT,

$\sin\theta = \frac{TB}{AT}$

$\therefore TB = AT \sin\theta$. 1

In ΔCAT, $\angle ATC = [180 - (\alpha + \beta)]$

$\therefore \frac{AT}{\sin\beta} = \frac{AC}{\sin(180 - (\alpha + \beta))}$

$\therefore AT = \frac{AC \sin\beta}{\sin(\alpha + \beta)}$ 1

$\therefore AT = \frac{d \sin\beta}{\sin(\alpha + \beta)}$ $\therefore TB = \frac{d \sin\beta \sin\theta}{\sin(\alpha + \beta)}$ 1 (3)

QUESTION FOUR

a) $v = 2t - 4$

(i) $a = 2 \text{ m/s}^2$ ✓

(ii) stationary when $t = 2$ ✓

$x = t^2 - 4t + c$

when $t = 0$ $x = 3$ $\therefore c = 3$

$x = t^2 - 4t + 3$ ✓

when $t = 2$ $x = 4 - 8 + 3$

$= -1 \text{ m}$ ✓

(iii) when $t = 0$ $x = 3$ } 4m

when $t = 2$ $x = -1$ } 1m

when $t = 3$ $x = 0$

particle travels 5 m in the first 3 seconds. ✓

(5)

b) $\ddot{x} = -4x$, when $t = 0$ $v = 2$ and $x = 5$

(i) $x = a\cos(2t + \alpha)$

$\dot{x} = -2a\sin(2t + \alpha)$

$\ddot{x} = -4a\cos(2t + \alpha)$

$= -4\left[a\cos(2t + \alpha)\right]$

$= -4x$ ✓

(ii) $T = \frac{2\pi}{2} = \pi$ ✓

(iii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4x$

$\therefore \frac{1}{2}v^2 = -2x^2 + c$ ✓

when $v = 2$, $x = 5$

$2 = -50 + c$

$\therefore c = 52$ ✓

$\therefore v^2 = -4x^2 + 104$

$v^2 = 4(26 - x^2)$,

$v^2 = n^2(a^2 - x^2)$

$\therefore a = \sqrt{26}$ ✓

(iv) $\dot{x} = -2a\sin(2t + \alpha)$

$\dot{x} = -2\sqrt{26}\sin(2t + \alpha)$ ✓

$-1 \leq \sin(2t + \alpha) \leq 1$

$\therefore$ max speed is $2\sqrt{26}$ m/s. ✓

(7)

(c) $\ddot{x} = 8x(x^2 + 4) = 8x^3 + 32x$

when $t = 0$ $x = 0$ and $v = 8$

(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 8x^3 + 32x$

$\frac{1}{2}v^2 = 2x^4 + 16x^2 + c$ ✓

When $x = 0$, $v = 8$ $\therefore c = 32$ ✓

$\therefore \frac{1}{2}v^2 = 2x^4 + 16x^2 + 32$

$v^2 = 4x^4 + 32x^2 + 64$

$= 4(x^4 + 8x^2 + 16)$

$= 4(x^2 + 4)^2$ ✓

$\therefore$ speed $= \sqrt{v^2} = 2(x^2 + 4)$

(ii) $\frac{dt}{dx} = \frac{1}{2(x^2 + 4)}$

(Note v is always positive. Particle starts with $v > 0$ and $\ddot{x} > 0$ + $v \neq 0$)

$\therefore t = \frac{1}{4}\tan^{-1}\frac{x}{2} + c$

when $t = 0$, $x = 0$ $\therefore c = 0$

$\therefore t = \frac{1}{4}\tan^{-1}\frac{x}{2}$ ✓

when $x = 2$, $t = \frac{1}{4}\tan^{-1} 1$ ✓

$= \frac{\pi}{16}$ seconds.

(5)

(17)

Alternative solution to 6 b (iii)

$x = a\cos(2t + \alpha)$

$\dot{x} = -2a\sin(2t + \alpha)$

when $t = 0$ $x = 5$ & $v = 2$

$5 = a\cos\alpha$ and $2 = -2a\sin\alpha$

$\cos\alpha = \frac{5}{a}$, $\sin\alpha = -\frac{1}{a}$

$\therefore \frac{25}{a^2} + \frac{1}{a^2} = 1$

$\therefore 26 = a^2$

$\therefore a = \sqrt{26}$